Solution of the General Helmholtz Equation in Homogeneously Filled Waveguides Using a Static Green's Function

Mahesh Balagangadhar, Tapan K. Sarkar, Jalel Rejeb, and Rafael R. Boix

***Abstract*— The new boundary-integral method used in this paper illustrates a novel approach to solve the general Helmholtz equation in homogeneously filled waveguides. Based on the method-of-moments Laplacian solution, the main feature of this formulation is that the Helmholtz equation is "reduced" to the Poisson's equation, which is then solved by using a static Green's function. In other words, the Green's function used in this method is frequency independent, unlike the most conventionally used Hankel functions. Hence, the computational time, while analyzing the waveguide over a range of different frequencies, is reduced considerably compared to other well-known numerical methods, since the frequency term just appears as a scaling factor in the evaluation of matrix elements. The numerical results obtained using the present method compare well with actual results (in the case of rectangular waveguides) and published results (in the case of L-shaped and single-ridge waveguides).**

***Index Terms*—Helmholtz equation, Green's function, waveguide.**

I. Introduction

The two-dimensional (2-D) Helmholtz equation appears in a variety of physical phenomena and engineering applications, such as heat conduction, acoustic radiation, and water wave propagation. In electromagnetics, the Helmholtz equation often appears as the governing equation for waveguide problems. Diverse numerical methods like the Ritz–Galerkin method, the surface integral-equation method, and the finite-element method have been employed to solve this equation.

In the Ritz–Galerkin method, which has been used in [4], [6]–[9], an integral equation for the transverse electric field at the inner apertures of the waveguides is established. The surface integral-equation approach, which has been used in [5], [12], and [17], starts from surface integral equations for the current–density in the waveguide walls for both TM and TE modes. The application of the method of moments to the solution of these integral equations in both methods leads to obtaining homogeneous systems of linear equations. The matrix coefficients of these systems of equations are given by infinite summation in the case of Ritz–Galerkin method and by integrals containing Hankel's functions, which must be numerically computed in case of surface integral-equation method. In both cases, the cutoff wavenumbers of the waveguides are obtained by solving iteratively nonlinear equations which arise when the determinant of the matrices of the systems of linear equations is set to zero (nontrivial solution condition). This consumes large central processing unit (CPU) time.

Manuscript received November 18, 1996; revised September 29, 1997. The work of R. R. Boix was supported by the Spanish Government, DGICYES, under Grant PR95-385.

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Publisher Item Identifier S 0018-9480(98)02022-5.

The finite-element method has been used in [3], [10], [11], [13]–[15]. In this method, the cutoff wavenumbers of hollow waveguides are usually obtained by using variational expressions which relate the propagation constants and the operating frequencies of the waveguides with the values of the electric and magnetic fields inside these waveguides. One constraint of the finite-element method is that most field components that are to be approximated have to be continuous in the cross sections of the waveguides since the variational expressions used in the computation of the cutoff wavenumbers involve derivatives of the mentioned fields components. Another constraint is that when the cross section of the waveguide to be analyzed presents reentrant corners (this happens with single- and double-ridged waveguides, L-shaped waveguides, crossed rectangular waveguides, etc.), the basis functions for the field components should take into account the singular behavior of these components at the conductor corners. This can be partly performed with simple basis functions (polynomials) by using a denser mesh of elements around the corners [10] or by using special basis functions such as functions possessing the exact singular behavior of the fields [11] or functions that permit the normal field to be discontinuous between adjoining elements [13]. However, although these latter basis functions have proved to be adequate to account for field singularities, they lead to matrices in the generalized eigenvalue problem whose coefficients have to be computed by means of numerical integration, thus increasing the CPU time requirements [11], [13]. It should also be said that when the effect of field singularities in reentrant corners is ignored when applying the finite-element method, very inaccurate results may be obtained (see [11, Sec. VI] and see the results for modified transverse (MT) equation in [15, Th. IV]).

In the new method [1] discussed in this paper, two integral equations for the axial components of the fields (electric and magnetic) inside a hollow waveguide are obtained by treating the Helmholtz equations for these axial components as Poisson's equations and by using a frequency-independent Green's function. The integral equations are solved based on the method-of-moments Laplacian solution, and it is proven in this paper that thanks to the use of a static Green's function, the computation of the cutoff wavenumbers for the modes propagating in the waveguide reduces to obtaining the eigenvalues of a matrix. The coefficients of the aforementioned matrix can be obtained in closed form when pulse basis functions and point matching are used to solve the integral equation.

In comparison with the Ritz–Galerkin method and the surface integral-equation method, the cutoff wavenumbers of the first modes are all obtained in one shot using this method by solving a relatively simple eigenvalue problem, whereas in any of the other two methods, the evaluation of each cutoff wavenumber requires one to iteratively obtain the roots of a nonlinear function which has to be obtained as the determinant of a matrix whose coefficients are not available in closed form. It is clear that the latter procedure takes much longer than the former. Moreover, the Green's function used in this method has only spatial dependence and is entirely frequency independent. This also greatly enhances the performance of the method in terms of computational time involved when the values of the propagation constants of different modes are to be obtained over different frequencies. The matrix elements are computed in far lesser time since the frequency dependence appears as only a scaling factor at each frequency.

By comparison with the finite-element method, the new method also presents a clear advantage. Since the unknown functions in the new method (axial components of the fields) are solutions of integral equations, the continuity of these functions in the cross section of the waveguides and the ability to handle field singularities at reentrant corners are not necessary conditions to be imposed on the basis functions employed in the approximation of the unknown functions. This is so because the integral of functions containing discontinuities and singularities (of the type shown by the fields at reentrant conductor corners) does not pose any problems. Therefore, the basis functions employed in the new method can be chosen to be simpler (e.g., pulse functions) than those employed in the finite-element method. An immediate consequence of this is that for a given accuracy, the size of the matrices involved in the application of the new method should be smaller than the size of the matrices involved in the application of the finite-element method. As a counterpart, the matrices related to the application of the finite-element method are sparse and those related to the application of the new method are not. The reliability of the new method in comparison with the finite-element method, and also the finite-difference time-domain method, has been discussed in [1] and [20].

The accuracy of the results obtained by this method is compared to actual results (for rectangular waveguides) and published results (for L-shaped and single-ridge waveguides), in this paper.

II. Solution of the General Helmholtz Equation in Homogeneously Filled Waveguides

In waveguides, solution of the Helmholtz equation determines the electromagnetic field configuration within the guides. It is convenient to divide the possible field configurations within the waveguides into two sets, namely TM waves and TE waves, each of which are governed by similar Helmholtz equations.

If we consider a waveguide in which the direction of propagation of the wave is along the z-direction, then the Helmholtz equations are as follows.

TM_z *Case* ($H_z \equiv 0$):

$$\nabla^2 E_z(x, y) + (\omega^2 \mu\epsilon - K_z^2) E_z(x, y) = 0 \tag{1}$$

with appropriate boundary conditions

$$E_z = 0, \qquad \text{on the conductor walls.}$$

TE_z *Case* ($E_z \equiv 0$):

$$\nabla^2 H_z(x, y) + (\omega^2 \mu\epsilon - K_z^2) H_z(x, y) = 0 \tag{2}$$

with appropriate boundary conditions

$$\partial H_z / \partial n = 0, \qquad \text{on the conductor walls}$$

where $\partial H_z/\partial n$ represents the normal derivative.

Here,

E_z z-component of the electric field;
H_z z-component of the magnetic field;
ω angular frequency $= 2\pi f$;

where

f frequency of interest;
μ permeability of the homogeneous medium;
ϵ permittivity of the homogeneous medium;
K propagation constant in the z-direction.

A solution to the general Helmholtz equation for a smooth function Ψ defined in a 2-D region $\mathcal{R}$, with contour $\mathcal{C}$ is

$$\nabla^2 \Psi(x, y) + \lambda(x, y)\Psi(x, y) = F(x, y) \tag{3}$$

where λ and F are known functions on the domain $\mathcal{R}$, discussed in [1].

On the contour $\mathcal{C}$, the boundary condition can be of Dirichlet, Neumann, or mixed type, as given by the general form

$$\alpha\Psi + \beta \frac{\partial \Psi}{\partial n} = \gamma \tag{4}$$

where α, β, and γ are known spatial functions. Further, $\partial\Psi/\partial n$ represents the normal derivative.

It is shown by Rejeb *et al.* [1], that the Helmholtz equation could be reduced to Poisson's equation

$$\nabla^2 \Psi(x, y) = -G(x, y) \tag{5}$$

with

$$G(x, y) = \lambda(x, y)\Psi(x, y) - F(x, y). \tag{6}$$

The solution is then expressed as

$$\Psi = \phi_h + \phi_p \tag{7}$$

where ϕ_h is the solution to the homogeneous Poisson's equation (Laplace's equation)

$$\nabla^2 \phi_h = 0 \tag{8}$$

and ϕ_p is the particular integral

$$\nabla^2 \phi_p = -G(x, y). \tag{9}$$

The potential ϕ_h can be assumed to be produced by some equivalent charges σ located on the contour $\mathcal{C}$, and can be obtained using the following integral, as explained in [2]:

$$\phi_h(x, y) = \frac{1}{2\pi} \int_{\mathcal{C}} \sigma(x', y') \cdot \ln\left(\frac{k}{\sqrt{(x - x')^2 + (y - y')^2}} \right) dl' \tag{10}$$

where l' is the arc length on the contour $\mathcal{C}$.

The particular solution of the Poisson's equation is given by

$$\phi_p(x, y) = \frac{1}{2\pi} \int\int_{\mathcal{R}} G(x', y') \cdot \ln\left(\frac{k}{\sqrt{(z - x')^2 + (y - y')^2}} \right) dx'\, dy'. \tag{11}$$

In (10) and (11), (x, y) and (x', y') denote the spatial coordinates of the field and source points, respectively, and k is an arbitrary constant which provides the potential at the reference point, and is taken to be 100 in our calculations.

Using the method of moments, involving pulse–expansion basis functions and point-matching techniques at the midpoints of the N discretized subregions of $\mathcal{R}$, and M subcontours of $\mathcal{C}$, the solution can be obtained from a system of matrix equations [1], given by

$$[A] \cdot [\Psi_i] = [B] \tag{12}$$

where

$$[A] = ([p_{ji}][l_{ji}]^{-1}[b_{ji}] - [q_{ji}])[\lambda_i] + [I] \tag{13}$$

$$[B] = [p_{ji}][l_{ji}]^{-1}[\gamma_j] + ([p_{ji}][l_{ji}]^{-1}[b_{ji}] - [q_{ji}])[F_i] \tag{14}$$

and $[I]$ denotes the $N \times N$ identity matrix. Here Ψ_i, F_i, and λ_i refer to the values of Ψ, F and λ, respectively, at the midpoints of the discretized subregions. The computation of matrices $[p_{ji}]$, $[q_{ji}]$, $[l_{ji}]$, and $[b_{ji}]$ have been discussed in [1]. An important inference in the computation of these matrices is that they do not involve the frequency term and, hence, the matrix elements remain unchanged even while solving the Helmholtz equation over different frequencies.

TABLE I

General Equation (3)	TM_z Equation (1)	TE_z Equation (2)
$\nabla^2\Psi + \lambda\Psi = F$	$\nabla^2 E_z + (\omega^2\mu\epsilon - K_z^2)\cdot E_z = 0$	$\nabla^2 H_z + (\omega^2\mu\epsilon - K_z^2)\cdot H_z = 0$
Ψ	E_z	H_z
λ	$\omega^2\mu\epsilon - K_z^2$	$\omega^2\mu\epsilon - K_z^2$
F	0	0
$\alpha\Psi + \beta\cdot\partial\Psi/\partial n = \gamma$	$E_z = 0$	$\partial H_z/\partial n = 0$
γ	0	0

Comparing (1) and (2) with (3), and also the boundary conditions of the TM_z and TE_z cases with that of the general equation, we can draw the following analogies from Table I.

Examining (14)

$$[B] = [p_{ji}][l_{ji}]^{-1}[\gamma_j] + ([p_{ji}][l_{ji}]^{-1}[b_{ji}] - [q_{ji}])[F_i].$$

It can be inferred that $[B] = 0$, since $F = 0$ and $\gamma = 0$ for TM_z and TE_z cases.

In the case of TM_z, (12) reduces to the form

$$[A]\cdot[E_{zi}] = 0. \tag{15}$$

In the case of TE_z, (12) reduces to the form

$$[A]\cdot[H_{zi}] = 0. \tag{16}$$

Here again, E_{zi} and H_{zi} refer to the values of E_z and H_z at the midpoints of the subregions of the discretized waveguide cross section.

For (15) and (16), nontrivial solutions exist for $[E_{zi}]$ and $[H_{zi}]$ only if the matrix $[A]$ is singular. The condition for nontrivial (i.e., nonzero) solutions to exist for $[E_{zi}]$ and $[H_{zi}]$ is

$$\det[A] = 0 \tag{17}$$

where $\det[A]$ stands for *determinant of* $[A]$. We know from (13) that

$$[A] = ([p_{ji}][l_{ji}]^{-1}[b_{ji}] - [q_{ji}])[\lambda_i] + [I]$$

and we also know that for TM_z and TE_z cases

$$\lambda = \omega^2\mu\epsilon - K_z^2. \tag{18}$$

Hence, given the frequency at which the Helmholtz equation is to be solved, $\det[A]$ would be a function of K_z, the roots of which give the values of K_z for which $\det[A] = 0$. Once these K_z values are known, the eigenvector of $[A]$ corresponding to the minimum eigenvalue gives the nontrivial solutions for $[E_{zi}]$ and $[H_{zi}]$ in case of TM_z and TE_z cases, respectively.

Once $[E_{zi}]$ and $[H_{zi}]$ are determined at the grid points, using Gaussian elimination for instance, the values of E_z and H_z at any other point can be obtained using ordinary matrix multiplications, as explained in [1].

As mentioned earlier, the main feature of this formulation is the use of a frequency-independent Green's function, which in this case is

$$\frac{1}{2\pi}\cdot\ln\left(\frac{k}{\sqrt{(x-x')^2+(y-y')^2}}\right).$$

Thus, while analyzing the waveguide over different frequencies, computation of matrix elements using this method is relatively less complicated and involves lesser computational time compared to other methods which make use of frequency-dependent Green's functions, e.g., Hankel functions. The frequency term just appears as a scaling factor in all the matrix computations involved in this method.

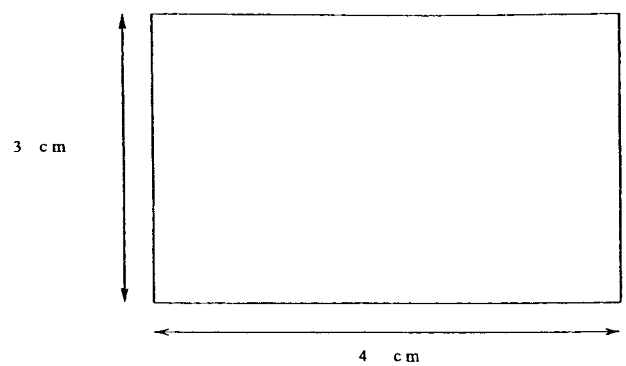

Fig. 1. Rectangular waveguide.

A. Calculation of Propagation Constants of Different Modes

It is evident that for the existence of nontrivial (nonzero) solutions for $[E_{zi}]$ and $[H_{zi}]$, it is necessary that (17) be satisfied.

Let us define a matrix $[Z]$

$$[Z] = ([p_{ji}][l_{ji}]^{-1}[b_{ji}] - [q_{ji}]). \tag{19}$$

Hence, (17) becomes

$$\det([Z][\lambda_i] + [I]) = 0 \tag{20}$$

which can be rewritten as

$$\det\left([Z] - \left(\frac{-1}{\lambda_i}\right)[I]\right) = 0. \tag{21}$$

Equation (21) is similar to the *characteristic equation* of matrix $[Z]$, with its eigenvalues given by $-1/\lambda_i$. Knowing that λ_i above $\equiv \omega^2\mu\epsilon - K_z^2$ for TM_z and TE_z cases, it can be concluded that

$$\frac{-1}{\lambda_i} = \frac{1}{(K_z^i)^2 - \omega^2\mu\epsilon} = EV_i^{[Z]}, \qquad i = 1, 2, \cdots, N. \tag{22}$$

where K_z^i is the propagation constant of the ith mode and $EV_i^{[Z]} \equiv$ ith eigenvalue of $[Z]$.

Equation (22) can be rearranged as

$$(K_z^i)^2 = \omega^2\mu\epsilon + \frac{1}{EV_i^{[Z]}}. \tag{23}$$

Therefore, the propagation constants of different modes in the waveguide are given by the following:

For $(K_z^i)^2 > 0$,

$$K_z^i = \sqrt{\omega^2\mu\epsilon + \frac{1}{EV_i^{[Z]}}} \quad \text{propagating modes} \tag{24}$$

For $(K_z^i)^2 < 0$,

$$K_z^i = j\sqrt{-\omega^2\mu\epsilon - \frac{1}{EV_i^{[Z]}}} \quad \text{nonpropagating modes.} \tag{25}$$

TABLE II
CUTOFF WAVENUMBERS FOR AIR-FILLED RECTANGULAR WAVEGUIDE

Mode No.	Mode	k_c actual(rad/cm)	k_c computed(rad/cm)	Diff. %
1 0.	TE_z	0.7857	0.7921	0.81
0 1.	TE_z	1.0476	1.0536	0.57
1 1.	TE_z , TM_z	1.3095	1.3239	1.00
2 0.	TE_z	1.5714	1.5827	0.72
2 1.	TE_z , TM_z	1.8886	1.9108	1.10
0 2.	TE_z	2.0952	2.1095	0.68
1 2.	TM_z , TM_z	2.2377	2.2610	1.00
3 0.	TE_z	2.3571	2.3896	1.30

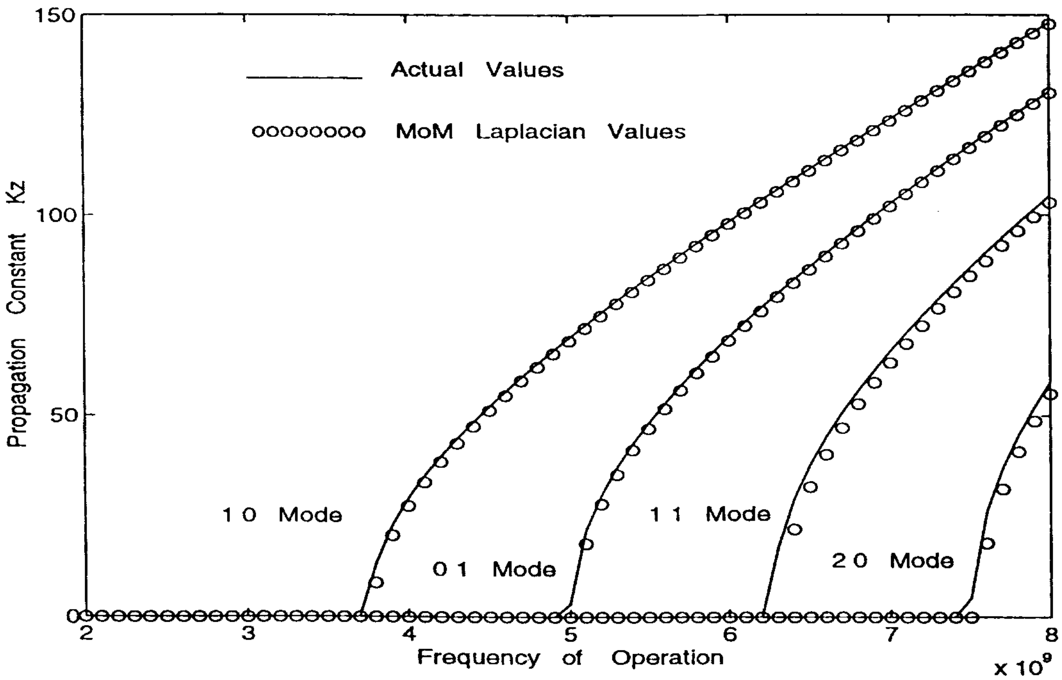

Fig. 2. Propagation characteristics in rectangular waveguide 4 × 3 cm.

Results of propagation constants of various modes in a rectangular waveguide computed by this method are shown in Section III, and compare well with actual results.

B. Calculation of Cutoff Frequencies for Different Modes

The cutoff frequencies for the various propagating modes in the waveguide are given by

$$f_c^i = \frac{v}{2\pi}\sqrt{(\omega^2\mu\epsilon - (K_z^i)^2)} \tag{26}$$

where $f_c^i \equiv$ cutoff frequency of the ith mode.

Here, $v \equiv$ velocity of light in the homogeneous medium $\equiv 1/\sqrt{\mu\epsilon}$. It can be deduced from (23) that

$$\omega^2\mu\epsilon - (K_z^i)^2 = \frac{-1}{EV_i^{[Z]}}, \qquad i = 1, 2, \cdots, N.$$

Using the above relation in (26), we find the cutoff frequencies for the first N propagating modes

$$f_c^i = \frac{v}{2\pi}\sqrt{\frac{-1}{EV_i^{[Z]}}}, \qquad i = 1, 2, \cdots, N. \tag{27}$$

The cutoff wave number k_c^i of the ith mode can be calculated from the cutoff frequency using the relation

$$k_c^i = \frac{2\pi f_c^i}{v}, \qquad i = 1, 2, \cdots, N.$$

This method thereby provides a straightforward approach to find the cutoff frequencies (and, hence, cutoff wavenumbers) of any waveguide structure without resorting to scanning over a wide range of frequencies, as is done in the Ritz–Galerkin and surface integral-equation methods. Results for the cutoff wavenumbers of rectangular, L-shaped, and single-ridge waveguides are given in Section III.

III. RESULTS

A. Rectangular Waveguide

A simple case of a waveguide is the rectangular waveguide. For the waveguide in Fig. 1, the region was divided into 100 subregions and the boundary was discretized into 96 subcontours. The maximum matrix size involved in computations was 100 × 100. Results have been displayed in Table II for the cutoff wave numbers of the first eight $\mathrm{TM}_z/\mathrm{TE}_z$ modes. The computational time involved in finding the cutoff wavenumbers of the first 100 modes on a Sun SPARC 10 workstation was 16 s.

Fig. 2 gives the values of propagation constants of the first four propagating modes in the rectangular waveguide shown in Fig. 1.

TABLE III
CUTOFF WAVENUMBERS FOR AIR-FILLED L-SHAPED WAVEGUIDE

Mode No.	Mode	k_c *published(rad/cm)*	k_c *computed(rad/cm)*	Diff. %
1.	TM_z	4.8190[19]	4.8657	0.97
2.	TM_z	6.1361[17]	6.2213	1.38
3.	TM_z	6.9908[17]	7.1151	1.77
4.	TM_z	8.5525[17]	8.7422	2.21
5.	TE_z	1.8800[4]	1.9010	1.10
6.	TE_z	2.9159[17]	2.9631	1.37
7.	TE_z	4.8755[17]	4.9720	1.97
8.	TE_z	5.2463[17]	5.3698	2.35

TABLE IV
CUTOFF WAVENUMBERS FOR AIR-FILLED SINGLE-RIDGE WAVEGUIDE

Mode No.	Mode	k_c *published(rad/cm)*	k_c *computed(rad/cm)*	Diff. %
1.	TM_z	12.1640[5]	12.2338	0.57
2.	TM_z	12.2938[17]	12.4106	0.95
3.	TM_z	13.9964[17]	14.2152	1.56
4.	TM_z	15.5871[17]	15.8221	1.50
5.	TE_z	2.2566[5]	2.2688	0.54
6.	TE_z	4.9436[17]	5.0149	1.44
7.	TE_z	6.5189[17]	6.6289	1.68
8.	TE_z	7.5642[17]	7.7097	1.92

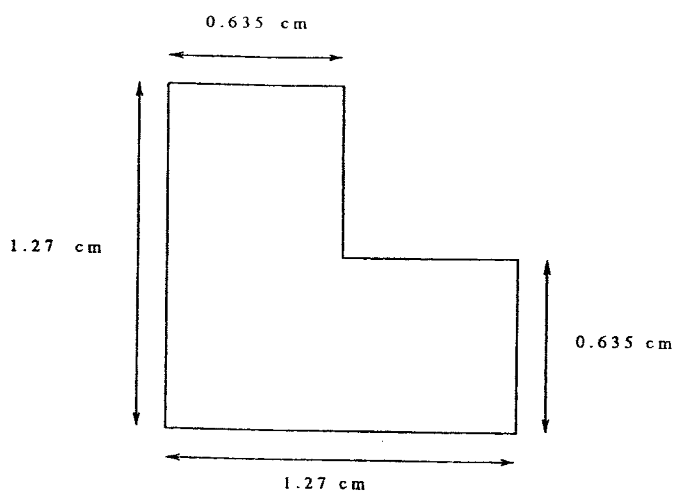

Fig. 3. L-shaped waveguide.

B. L-Shaped Waveguide

The first four TM_z and TE_z mode cutoff wavenumbers were computed for the L-shaped hollow waveguide shown in Fig. 3. Since analytical results are not available for this waveguide, the obtained results in Table III were compared with published data. For the waveguide above, the region was divided into 108 subregions and the boundary was discretized into 96 subcontours. The maximum matrix size involved in computations was 108×108. The computational time involved in finding the cutoff wavenumbers of the first 108 modes in each case on a Sun SPARC 10 workstation was 18 s.

C. Single-Ridge Waveguide

A single ridge waveguide is a popular means of getting higher bandwidth. The first four TM_z and TE_z mode cutoff wavenumbers

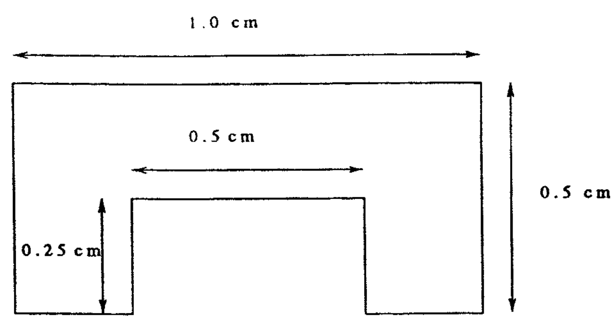

Fig. 4. Single-ridge waveguide.

were computed for the single-ridge hollow waveguide shown in Fig. 4. Results have been displayed in Table IV and compared with published data. For the waveguide, the region was divided into 96 subregions and the boundary was discretized into 112 subcontours. The maximum matrix size involved in computations was 96×96. The computational time involved in finding the cutoff wavenumbers of the first 96 modes in each case on a Sun SPARC 10 workstation was 18 s.

IV. CONCLUSION

The method discussed in this paper presents a very efficient technique based on the method-of-moments Laplacian solution to solve the general Helmholtz equation in homogeneously filled waveguides. In addition to "reducing" the Helmholtz equation to the Poisson's equation, the main feature of this method is the use of a frequency-independent Green's function, which considerably reduces the computational time and complexity involved in the evaluation of matrix elements while solving the Helmholtz equation over a range of different frequencies. The numerical results obtained using the present method compare well with actual results (in the case of rectangular waveguides) and published results (in the case of L-shaped and single-ridge waveguides). As the next step, the application of the same kind of formulation for solving the general Helmholtz equation in partially filled (inhomogeneous) waveguide structures is presently being studied.

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